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Las Paradojas Matemáticas en el proceso de enseñanza - aprendizaje

Mathematical Paradoxes in the teaching - learning processe

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Resumen: En este trabajo se exponen criterios que fundamentan la idea de utilizar las Paradojas Matemáticas en el proceso de enseñanza – aprendizaje de la Matemática, se presentan como fuente de asombro, curiosidad y motivación en los estudiantes para así despertar el interés por el estudio y la búsqueda de información para lograr una cultura general integral. Los autores proponen el análisis metodológico de algunas de estas paradojas como ejemplo, y los resultados de los experimentos realizados durante la investigación en dos años, con estudiantes de varios niveles de enseñanza.

Palabras clave: Paradojas; Matemática; Demostración; Motivación.

Abstract: In this work, criteria are presented that support the idea of using Mathematical Paradoxes in the teaching-learning process of Mathematics, they are presented as a source of amazement, curiosity and motivation in students in order to awaken interest in study and search of information to achieve a comprehensive general culture. The authors propose the methodological analysis of some of these paradoxes as an example, and the results of the experiments carried out during the investigation in two years, with students of various levels of education.

Keywords: Paradoxes; Math; Demonstration; Motivation.

Introduction

The theme of paradoxes in Mathematics in the teaching-learning process has been dealt with in several works, among them (Talero, Santana, & Mora, 2015) which, based on the graphic interpretation of the uniform rectilinear movement and the geometric series, clarify the popular paradox of Achilles and the turtle. In Physics, this result can be used as an example in elementary courses of calculus and mechanics. In the work of (Tasenm, 2010) the history of Mathematical Paradoxes and the first mathematicians who began to study this topic from the point of view of mathematical logic are briefly recounted. (Macho, 2012) makes a tour through several concrete examples of paradoxes of different types, such as geometric disappearances, anamorphosis, logical paradoxes, of infinity, of vagueness, among others.

. In (Martínez, 1999) a sequence of mathematical reasoning based on the resolution of a paradox is presented, with the intention that this proposal be used in the training courses of secondary school mathematics teachers. The author proposes that by means of the realization of formative modules based on the search of meaning and resolution of paradoxes, it is possible to put the teachers in formation to make mathematics with elementary problems, idea that later will make arrive to their students.

More information about the theory and applications of Mathematical Paradoxes can be found in (Acharya, Basu, & Das, 2015), (Castro & Hernando, 2003) and (Chaitin, 2003)

Currently, one of the biggest problems facing academic science around the world is the lack of motivation of students towards it, which goes hand in hand with little interest in their study and non-participation in classes. The most serious and worrying case is that of Mathematics, a problem from which few people are unaware. Cuba is not exempt from this situation, and it is always a goal to get students in the classroom motivated to study something they don't like, and therefore, many times it is a matter of getting them to like it first. In this sense, in this work, the authors propose as an objective to approach the subject of Mathematical Paradoxes, proposing them as one more tool to support teachers in the mission of motivating students, taking as a fundamental problem, the students' lack of motivation towards this science. In addition, the paradoxes are proposed as a way to generate problematic situations that make possible the debate and the generation of ideas among students. On the other hand, the results of the experiments developed in several groups of different teaching levels in several schools are presented.

Development

Some definitions and types of paradoxes are presented, which will help the reader to understand better. Then some practical examples of paradoxes will be presented and the solutions of some will be seen. It is interesting to note that there are quite a few types of paradoxes, as well as definitions of them, so the authors do not claim to be exhaustive.

Types of paradoxes

The term "paradox" can be interpreted as something that at first sight seems to be false but is in fact true, or that seems to be true but is strictly speaking false, or simply that it contains contradictions in itself. The term paradox originates from the Latin paradoxus, which in turn comes from the Greek paradoxos -παράδοξος-. Etymologically, paradoxos means what parà ten doxan, "that which goes against public opinion" (Tasenm, 2010). According to (Muñoz, 2007) the paradoxes can be classified as follows:

According to their veracity: In the first group, one can find paradoxes that only seem to be so (what they say is really true or false), others contradict themselves, while others depend on their interpretation to be paradoxical or not.

1.1. True Paradoxes

These are results that seem perhaps absurd in spite of their demonstrable veracity. To this category belong most of the mathematical paradoxes.

- Birthday Paradox: What is the probability that two people in a meeting have the same birthday?
- Galileo's paradox: even though not all numbers are square numbers, there are no more numbers than square numbers.
- Infinite hotel paradox: an infinite room hotel can accept more guests, even if it is full.

1.2. Anthinomies

They are paradoxes that achieve a result that is self-contradictory, correctly applying accepted modes of reasoning. They show flaws in a previously accepted mode of reasoning, axiom or definition. For example, Grelling-Nelson's Paradox points out genuine problems in our understanding of ideas of truth and description. Many of them are specific cases, or adaptations, of Russell's Paradox.

- Russell's Paradox: Is there a set of all sets that are not
  contain themselves?

- Curry Paradox: "If I am not mistaken, the world will end in ten days"
- Liar's paradox: "This sentence is false"
- Grelling-Nelson paradox: Is the word "heterological," which means "not

describes itself", heterological?

1.3. Definition antinomies

These paradoxes are based on ambiguous definitions, without which they do not reach a contradiction.

- Paradox sorites: At what point does a pile stop being a pile when you remove grains of sand?
- Theseus paradox: When all the parts of a boat have been replaced, is it still the same boat? (pp. 4-6)

Another important group of paradoxes are the logical-mathematical ones. These are taken in many cases as the guiding light of other definitions. They constitute an important group and have contributed most to the foundations of Mathematics. Among them is the paradox of Banach-Tarski or Hausdorff, which, in a non-formal language, states that it is possible to make a three-dimensional puzzle out of a total of eight pieces, which, combined in a certain way, would form a complete and filled sphere (without holes) and, combined in another way, would form two filled spheres (without holes) of the same radius as the first one (Muñoz, 2007). This paradox has a purely mathematical foundation. It belongs to Topology and although it does not seem so, more than a paradox, it is a theorem with interesting consequences in the area.

There are other paradoxes whose center is in the concept of infinity like Galileo's: although not all numbers are square numbers, there are no more numbers than square numbers. It refers to natural (or integer) numbers. He states that the set of natural (or integer) numbers has the same amount of elements as the set of the square of its elements, although it seems that the latter has fewer numbers because it is a subset of the former. There are also Zeno's three paradoxes about movement: a) that of Achilles and the Turtle, which affirms that the first can never reach the turtle; b) that of the throwing of the arrow, affirms that the arrow never moves, in each instant it is always at rest; c) finally the paradox that affirms the impossibility of moving from one point A to another point B.

There are many other types of paradoxes, among them, Muñoz (2007) also highlights:

Paradoxes on Logic: Although all paradoxes are considered to be related to logic, there are some that directly affect its traditional bases and postulates. The most important paradoxes directly related to the area of logic are the antinomies, such as Russell's paradox, which show the inconsistency of traditional mathematics. In spite of this, there are paradoxes that do not contradict each other and that have helped to advance concepts such as proof and truth.

- Paradox of the King of France: is it true a statement about something that does not exist?
- Crow (or Hempel's crows) paradox: a red apple increases the probability that all crows are black. (p. 7)

Examples of paradoxes and their methodological treatment

The first paradoxes will be about false demonstrations, the last ones will be geometric. It will be proposed methodologically how teachers can deal with this issue with students. We will start with the demonstration that "2 = 1", the reasoning is as follows:

Let's start by stating that, a = b,

Multiplying by both members you have, a2 = ab,

Subtracting b2, a2 - b2 = ab - b2,

Factoring, (a + b) (a - b) = b(a - b),

Simplifying a - b, a + b = b,

By doing a = b (the initial assumption) you have, 2b = b,

And finally by simplifying b, we arrive at, 2 = 1.

Now, the main problem is that the final result is known to be false, but the idea then is where is the error in the procedure? what was done wrong? Apparently nothing, all the steps seem correct. At this point it is necessary to motivate the student to look for where the procedure could have failed, otherwise he will be accepting that 2 = 1, which does not seem to be so false now, in particular this implies that all the numbers are the same, and the demonstration could be left for independent study. The error is in simplifying a - b, in the 5th step, since it was assumed that a = b, therefore, a - b = 0, which means that it was divided by zero, which is not possible and that is why the error was reached.

Another interesting demonstration is that 1 is less than 0 ($1 < 0$):

We begin by stating that, $0.5 < 1$, which is true,

We apply logarithm in base 10 to both members, $\log 0.5 < \log 1$,

Dividing by log 0.5 both members and simplifying you have, ,

As $\log 1 = 0$, substituting you have, ,

Which is equivalent to, $1 < 0$.

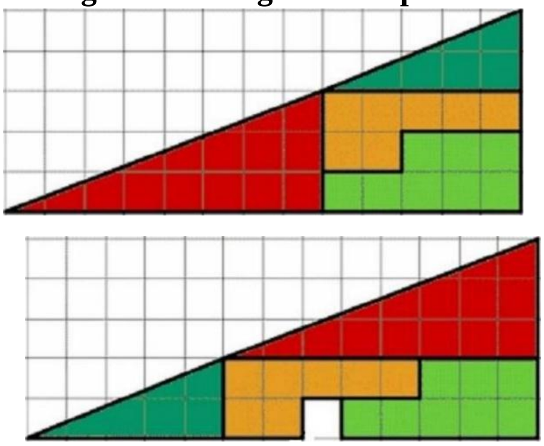
Again one arrives at the questions of before, making seemingly correct transformations, one arrives at a result, which, according to intuition, should be false. These are examples where it is reflected that it is not enough to make steps that seem correct, but the student must understand that, in the daily life, it is really necessary to verify if the steps are really correct analyzing the details. All the rules that have been used have been correct, and it was started from something true, where the error can be. These are the moments of problematic situation that must be taken advantage of so that the students participate, even the most backward one knows that there is a problem which he understands, that 1 cannot be less than 0, however, he himself does not know why that happens, situation that motivates him to, at least, think inside himself in looking for a solution.

Finally, after a thorough search, you can see that the error in these procedures is in the division by log 0.5, since it is a negative number, $\log 0.5 \approx -0.301$, and therefore, when dividing, you must change the sign from <, to >, which was not done, because log 0.5 hides the sign from being negative. This situation is common in many paradoxes, the fact that the solution is not complicated, but, is hidden and the complicated thing is to find it.

These simple solutions, such as changing the sign of an inequality, or division by zero, do not cease to be educational in themselves, since they remind the student to be careful in his calculations and in the procedures he performs, so as not to make mistakes like these. Finally, two interesting geometrical paradoxes will be exposed, whose solutions will not be explained for reasons of space.

The first paradox is presented below in figure 1. The first image presents a triangle divided into parts, in the second, these parts are rearranged in the same location as the area occupied by the original triangle. As a result, the pieces have to occupy the same space, however, it can be noticed that one of the squares that make up the triangle has disappeared, that is, area has been lost. This should not happen, since the same pieces that were cut were only rearranged, so they should form the same triangle, but this is not the case.

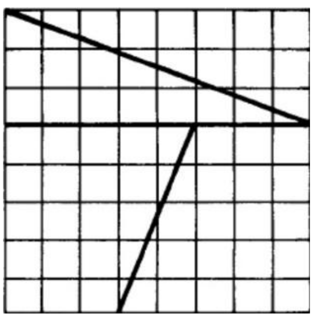
Figure 1. First geometric paradox

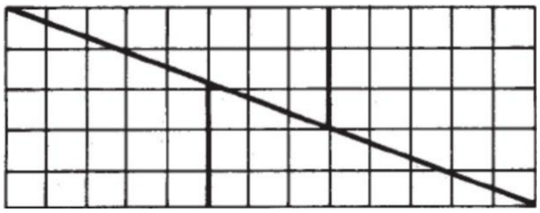


Source: (Male, 2012)

This paradox is curious and contradicts common sense, it is one of the most effective in the debate of students in Geometry classes. Very similar, is the following one, whose solution is based on the same principle as the one explained above. It can be seen in figure 2.

Figure 2. Second geometric paradox





Source: (Macho, 2012)

The first square in figure 2 is 8x8 squares, or 64. This square is cut into 4 parts, which are rearranged, like the paradox in figure 1, to form the rectangle below. The problem is that this rectangle is 5x13, which is equivalent to 65 squares, one more than the square that the pieces originally formed, which should not happen.

These are some examples of paradoxes that can be treated in class and whose solution the student can understand without advanced knowledge of Mathematics. There are many others, such as demonstrating that $1 = -1$, another that $2 = 1$, etc., which can fulfill the same objective of surprising and generating participative debate of the students in the classes. This debate is interesting, since the student sees himself in the need to think, but it is not a forced thought, but, motivated by himself, his belief that such situations cannot be given and the need to look for a solution to them. This process achieves that the student activates the brain, something that with independent studies and practical classes is desired to achieve, but that many times is not achieved.

For example, activities such as solving defined or undefined integrals, calculating derivatives, own values and vectors, making theoretical demonstrations, Geometry exercises, among others, put the student in the situation that he has to solve them because he is going to be evaluated, and he has to acquire that knowledge. However, in a paradox, even if he is not oriented to solve it, he himself enters the conflict that something is contradicting his common sense and what he knew as true until now, and he does not know why, and that is a reason for him to start trying to find the answer. An interesting point is that many times it is more difficult to see the solution of a paradox, than to solve class exercises.

The authors believe that the Mathematical Paradoxes, due to their characteristics, constitute a source to achieve amazement, curiosity and motivation in students towards the subjects of this science. They open the way to the interest for the study and the search of information. If they are well used in the teaching-learning process, it is also possible to achieve a comprehensive general mathematical culture.

Experiment and results

The experiment developed in research during two years (from September 2017 to May 2019) in Guantánamo, was applied to several schools, such as the "Celia Sánchez Manduley" Basic Secondary School in Bayate Arriba, El Salvador, the "Manuel Ascunce Domenech" of the Guantánamo municipality, the IPVCE "José M. Maceo Grajales", some groups of 2nd and 3rd year students of Economics, Accounting, Computer Science, Industrial and Mathematics of the University of Guantánamo. In general, a total of 310 students were experimented with.

The experiment consisted, first of all, in observing the opinion and attitude of the students in the classes, asking them oral and written questions, and conversing with them. The main idea is to do this without them noticing that they are being evaluated, so that they do not intentionally reflect a partial criterion for or against the Mathematics subjects. After knowing the general criteria of the students, the second phase is to include in the classes some topics related to the Mathematical Paradoxes, in search of motivation (this is done again with discretion, as something natural in the classes so that the results are impartial). The third and final part, is to compare the students' initial criteria, with those obtained after talking about several paradoxes.

As a result, initially 30% (93 students) of the total were motivated, after the experiments, about 63.87% (198 students) showed more interest and motivation in the classes, which corresponds to the initial 93 plus another 105 students. The remaining 36.13% of the students showed disinterest for various reasons. On the other hand, it was observed that, when posed as a problematic situation, almost all students participate in the search for the solution to try to solve the situation. Although this first test for two years did not show 100% approval as would have been desired, it did show effectiveness, and makes it clear that little by little, doing a continuous work in the teaching-learning process from the first teachings, with this and other tools, it is possible to achieve that the students in a general way, accept a little more this science.

In the case of measuring the criterion on student motivation, in the first and second phase, the authors decided to do it this way, precisely to seek an opinion as natural as possible. With methods such as a survey and others that numerically reflect the state of opinion, students would be too involved, other factors would enter into the result, and precisely, this is what is not wanted.

Conclusions

Paradoxes are part of the mathematical world and have contributed to the development of this science, they are a source of motivation and curiosity. This aspect is interesting to use in the process of teaching - learning, with the objective of looking for motivation in the students.

In this work, some paradoxes were developed, to serve as initiation and base for the teachers in their classes. Besides, the results of the research developed by the authors during two years were shown, where it is reflected the effectiveness of including topics related to the Mathematical Paradoxes in the classes in search of motivation and participation.

In future investigations it is interesting to continue approaching other topics related to the paradoxes that appear in each one of the Mathematics subjects, and to establish teaching - learning strategies to include these paradoxes in a natural way in the classes, leaving tasks, problematic situations of debate, among others, without this constituting delay or deviation in the contents of the program of the subject, but rather, complementing them.

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